

FOLIATION C^* -ALGEBRAS ON MULTIPLY FIBRED MANIFOLDS

ROBERT YUNCKEN

ABSTRACT. Motivated by index theory for semisimple groups, we study the relationship between the foliation C^* -algebras on manifolds admitting multiple fibrations. Let $\mathcal{F}_1, \dots, \mathcal{F}_r$ be a collection of smooth foliations of a manifold \mathcal{X} . We impose a condition of local homogeneity on these foliations which ensures that they generate a foliation \mathcal{F} under Lie bracket of tangential vector fields. We then show that the product of longitudinal smoothing operators $\Psi_c^{-\infty}(\mathcal{F}_1) \cdots \Psi_c^{-\infty}(\mathcal{F}_r)$ belongs to the C^* -closure of $\Psi_c^{-\infty}(\mathcal{F})$. An application to noncommutative harmonic analysis on compact Lie groups is presented.

2000 MSC Codes: 58J40 (primary), 22E30, 43A85 (secondary).

Keywords: pseudodifferential operators; foliation algebras; Lie groups; flag varieties.

1. INTRODUCTION

A major thread throughout index theory is the study of longitudinal pseudo-differential operators—operators which (pseudo)differentiate along the leaves of a foliation. This stream of ideas was already begun in the foundational papers of Atiyah and Singer [AS71] (for fibred manifolds), and generalized radically by Connes [Con82] (for foliations). However, to date all of the literature has concentrated on manifolds with a single foliation. The purpose of this paper is to begin to study foliation algebras on manifolds with multiple foliations.

It is important to understand the motivation here, which comes from very specific examples in representation theory. Recall that for semisimple Lie groups, much of the representation theory centres on the generalized principal series representations, which act on section spaces of line bundles over the flag manifold $\mathcal{X} := \mathbf{G}/\mathbf{B}$ of the group \mathbf{G} . Intertwining operators between these representations occur naturally as pseudodifferential operators along the various fibrations of \mathcal{X} .

It is the analysis of these operators that we are really interested in. Our decision to use the language of foliations rather than fibrations in this paper was made only to simplify the statements and proofs of the results, not to anticipate generalizations away from these fundamental examples.¹

In [Yun11], we demonstrated how the analysis of these operators can be applied to index theory. Specifically, we used the BGG-complex of $\mathrm{SL}(3, \mathbb{C})$ to construct an explicit model for Kasparov's γ element as the image of an element of the equivariant K -homology of the flag variety $K_G(\mathcal{X})$. This parallels earlier constructions by Kasparov and Julg ([Kas84, JK95, Jul02]) which were used to prove the Baum-Connes conjecture for all discrete subgroups of rank one Lie groups². The conjecture remains open in rank greater than one.

Central to the above mentioned construction of γ for $\mathrm{SL}(3, \mathbb{C})$ was a compactness theorem for products of negative order pseudodifferential operators along the

2000 *Mathematics Subject Classification.* 58J40 (primary); 22E30, 43A85 (secondary).

¹Having said that, one could certainly conceive of applications to parabolic geometries, following [ČSS01].

²The case of $\mathrm{Sp}(n, 1)$ was first proven in [Laf02] using slightly different methods.

fibrations of \mathcal{X} ([Yun10]). The proof used some highly nontrivial computations in noncommutative harmonic analysis for the maximal compact subgroup $SU(3)$. While that argument could, in principle, be generalized to all compact semisimple groups, in practice the computations become overwhelming.

In this paper, we change our approach by using noncommutative harmonic analysis in the sense of M. Taylor [Tay84]. This allows results to be proven in broad generality—in particular for any generalized flag manifold. Moreover, the proofs become considerably more enlightening.

Let us state the main results. Let \mathcal{X} be a smooth manifold equipped with a collection of r smooth foliations $\mathcal{F}_1, \dots, \mathcal{F}_r$. Let $\Psi_c^{-\infty}(\mathcal{F}_j)$ denote the set of longitudinally smoothing operators along \mathcal{F}_j with compact support. We shall explain exactly what we mean by this in Section 2. With the appropriate definition, these act as bounded operators on $L^2\mathcal{X}$, and their norm-closure $\overline{\Psi_c^{-\infty}(\mathcal{F}_j)}$ is a C^* -algebra. It contains the order $-d$ longitudinal pseudodifferential operators $\Psi_c^{-d}(\mathcal{F}_j)$ for any $-\infty \leq -d < 0$.

We shall control the relative geometry of the foliations by assuming the following local homogeneity condition. Let G be a connected Lie group, and let $\mathfrak{h}_1, \dots, \mathfrak{h}_r$ be a collection of Lie subalgebras of its Lie algebra \mathfrak{g} . We let \mathcal{H}_j denote the foliation of G generated by left translates of \mathfrak{h}_j .

Definition 1.1. The family of foliations $\mathcal{F}_1, \dots, \mathcal{F}_r$ will be called *locally homogeneous* (with structural data $\mathfrak{h}_1, \dots, \mathfrak{h}_r \leq \mathfrak{g}$) if there is an atlas of local charts $\phi_\alpha : U_\alpha \rightarrow \mathcal{X}$ with $U_\alpha \subseteq G$ such that $d\phi_\alpha$ maps each \mathcal{H}_j isomorphically to \mathcal{F}_j on its domain.

This condition ensures that the set of vector fields generated by $C^\infty(T\mathcal{F}_1), \dots, C^\infty(T\mathcal{F}_r)$ via Lie brackets is itself the space of vector fields tangent to a foliation, which we denote by \mathcal{F} . In each chart, $T\mathcal{F}$ is the bundle of left translates of the Lie algebra generated by $\mathfrak{h}_1, \dots, \mathfrak{h}_r$.

Theorem 1.2. Let $\mathcal{F}_1, \dots, \mathcal{F}_r$ be a locally homogeneous family of foliations. With \mathcal{F} as above,

$$\overline{\Psi_c^{-\infty}(\mathcal{F}_1)} \cdots \overline{\Psi_c^{-\infty}(\mathcal{F}_r)} \subseteq \overline{\Psi_c^{-\infty}(\mathcal{F})}.$$

Thus, successive smoothing along the directions of $\mathcal{F}_1, \dots, \mathcal{F}_r$ yields an operator which is almost smoothing, not just in the directions spanned by the $T\mathcal{F}_j$, but in all directions generated from them via Lie brackets.

If \mathcal{F} is the foliation of \mathcal{X} by a single leaf then $\overline{\Psi_c^{-\infty}(\mathcal{F})} = \mathcal{K}(L^2\mathcal{X})$. Thus, we have the following important corollary.

Definition 1.3. We shall say the foliations $\mathcal{F}_1, \dots, \mathcal{F}_r$ satisfy *Hörmander's condition* if the Lie algebra of all smooth vector fields on \mathcal{X} is generated by $C^\infty(T\mathcal{F}_1), \dots, C^\infty(T\mathcal{F}_r)$.

For a locally homogeneous family of foliations, this is equivalent to requiring that $\mathfrak{h}_1, \dots, \mathfrak{h}_r$ generate \mathfrak{g} as a Lie algebra.

Corollary 1.4. Let $\mathcal{F}_1, \dots, \mathcal{F}_r$ be a locally homogeneous family of foliations which satisfy Hörmander's condition. If $A_j \in \Psi_c^{-1}(\mathcal{F}_j)$ for each j , then the product $A_1 \cdots A_r$ is a compact operator.

Applications to equivariant index theory require further results, which we shall defer to a subsequent paper. For now, we will provide a brief application to noncommutative harmonic analysis on compact Lie groups.

Suppose that K is a compact Lie group, and K_1, K_2 are closed subgroups which generate K . Let U be a unitary representation of K on a Hilbert space H for which all irreducible K -types have finite multiplicity. If π_1, π_2 are irreducible representations for K_1 and K_2 , respectively, then the π_1 - and π_2 -isotypical subspaces of H are essentially orthogonal, in the sense that they have arbitrarily small inner products on the complement of some finite-dimensional subspace. (See Theorem 6.4 for a precise statement.)

The paper is organized as follows. In Section 2 we provide some background on noncommutative microlocal analysis as it pertains to longitudinal pseudodifferential operators on Lie groups. The technical heart of the paper is Section 3, in which we prove results about integral operators in nonsingular coordinate systems. In Sections 4 and 5 we apply this analysis to longitudinal pseudodifferential operators. The final section describes the above-mentioned application to noncommutative harmonic analysis.

We would like to thank N. Higson, C. DeBord and J.-M. Lescure for helpful conversations.

1.1. Notation. Throughout, G will denote a connected Lie group, and H_1, \dots, H_r closed subgroups. For any Lie group, we shall use the corresponding Fraktur letter to denote its Lie algebra, often without mention. We will denote left Haar measure on G by μ_G or $d_G x$, and the modular function by Δ_G .

2. LONGITUDINAL PSEUDODIFFERENTIAL OPERATORS

Let \mathcal{F} be a smooth foliation of a manifold \mathcal{X} . If \mathcal{F} is the tangent bundle to a smooth fibration $p : \mathcal{X} \rightarrow \mathcal{Y}$ then elements of $\Psi^d(\mathcal{F})$ can be defined as smooth families of pseudodifferential operators of order d on the fibres, as in [AS71]. Then $\Psi_c^d(\mathcal{F})$ will be the subset of those whose distributional kernel has compact support in $\mathcal{X} \times \mathcal{X}$. However, in what follows it will be convenient if we don't have to continually assume that our foliations are fibrations. For this, some brief technical remarks are in order to clarify our definitions.

As Connes observed, the correct home for pseudodifferential operators on foliations is the holonomy groupoid $\mathcal{G} := \mathcal{G}(\mathcal{X}, \mathcal{F})$. (See [Con94, Con82] for the definitions.) The smooth convolution algebra $C_c^\infty(\mathcal{G})$ is naturally represented on the L^2 -spaces of the leaves, and the resulting C^* -algebra is Connes' foliation algebra $C^*(\mathcal{X}, \mathcal{F})$.

However, in order to make sense of statements such as Theorem 1.2 above, we need the various convolution algebras to be all represented on a common Hilbert space. Thus, in this paper, we will be working with what J. Roe [Roe87] refers to as the "global representation" of $C_c^\infty(\mathcal{G})$ on $L^2(\mathcal{X})$. Specifically, for $k \in C_c^\infty(\mathcal{G})$, $f \in L^2(\mathcal{X})$, set

$$k \cdot f(x) := \int_{\mathcal{G}_x} k(\gamma) f(r(\gamma)) d\gamma.$$

where, r, s are the range and source maps of \mathcal{G} and $\mathcal{G}_x := s^{-1}(x)$. It is the image of this representation in $\mathcal{B}(L^2 \mathcal{X})$ which we refer to as $\Psi_c^{-\infty}(\mathcal{F})$. Note that this will not generally extend to a representation of $C^*(\mathcal{G})$, unless the transverse component of Riemannian measure is holonomy invariant (as is the case for fibrations).

In any case, we denote by $\overline{\Psi_c^{-\infty}(\mathcal{F})}$ the norm-closure of $\Psi_c^{-\infty}(\mathcal{F})$ in $\mathcal{B}(L^2(\mathcal{X}))$. If \mathcal{F} comes from a fibration, $\overline{\Psi_c^{-\infty}(\mathcal{F})} \cong C_r^*(\mathcal{X}, \mathcal{F})$.

We also remark that $\Psi_c^{-d}(\mathcal{F})$ is dense in $\overline{\Psi_c^{-\infty}(\mathcal{F})}$ for any $-\infty \leq d < 0$. Thus, our C^* -algebraic approach destroys any notion of order of (pseudo-) differentiation and hence precludes all of the subtle analytic estimates that a full

pseudodifferential calculus affords. On the other hand, this norm-density allows us to work with the relatively simple class of longitudinally smoothing operators, and still our results remain strong enough for applications to index theory, which is concerned with much coarser analytic properties (Fredholmness, compactness, *etc*).

We now specialize to Lie groups, where we will reformulate longitudinal smoothing operators following the pattern of [Tay84].

Let G be a connected Lie group, and let H be a connected Lie subgroup, not necessarily closed. We endow H with the topology associated to its intrinsic smooth structure, not the subspace topology. Let \mathcal{H} denote the foliation of G by left-cosets of H .

A longitudinally smoothing operator $K \in \Psi_c^{-\infty}(\mathcal{H})$ is given by an integral formula

$$(2.1) \quad Ku(x) := \int_H k(x, h)u(h^{-1}x) dh,$$

where $k \in C_c^\infty(G \times H)$. We will sometimes denote such an operator by $\text{Op}_H k$.

Define $L^2(G)$ with respect to left-invariant Haar measure. The adjoint of $\text{Op}_H k$ is $\text{Op}_H(k^*)$ where

$$(2.2) \quad k^*(x, h) := \overline{k(h^{-1}x, h^{-1})} \Delta_H(h).$$

3. INTEGRAL OPERATORS IN SINGULAR COORDINATE SYSTEMS

Let G, H be as in the previous section. We need to generalize the operators of (2.1) by reparameterizing the variable $h \in H$ with a singular change of coordinates, as follows.

Let M be a smooth manifold with smooth measure dm and let $\phi : M \rightarrow H$ be a smooth function. We consider operators A of the form

$$(3.1) \quad Au(x) := \int_M a(x, m)u(\phi(m)^{-1}x) dm$$

for $a \in C_c^\infty(G \times M)$.

Lemma 3.1. *The formula (3.1) defines a bounded operator A on $L^2(G)$ with norm $\|A\| \leq \|a\|_\infty \text{Vol}(M\text{-supp}(a))$.*

Proof. Let L_g denote the left regular representation of $g \in G$ on $L^2 G$. We can write $A = \int_M A_m dm$ where A_m is the operator $u \mapsto a(\cdot, m)L_{\phi(m)}u$. Note that $\|A_m\| \leq \|a\|_\infty$, and that $A_m = 0$ for $m \notin M\text{-supp}(a)$. The result follows. \square

Define the *critical set* of $\phi : M \rightarrow H$ as $C(\phi) := \{m \in M \mid D\phi(m) \text{ is not onto}\}$. The M -support of $a \in C_c^\infty(G \times M)$, denoted $M\text{-supp}(a)$, is the projection of the support of a onto M .

Lemma 3.2. *Suppose that $M\text{-supp}(a) \cap C(\phi) = \emptyset$. Then the operator A of (3.1) is in $\Psi_c^{-\infty}(\mathcal{H})$.*

Proof. Let $d := \dim(H)$ and $n := \dim(M)$. Note that if $n < d$ then $C(\phi) = M$, so necessarily $n \geq d$. By using a partition of unity subordinate to local charts on M , we may reduce to the case where M is a bounded open subset of \mathbb{R}^n .

Let $N := \binom{n}{d}$. Let E_1, \dots, E_N denote the coordinate d -planes of \mathbb{R}^n (in any order). For each of these, we will define a Jacobian of ϕ in the spirit of the Implicit Function Theorem. Thus, let p_i denote the orthogonal projection of \mathbb{R}^n onto E_i^\perp , and augment ϕ to the map

$$\Phi_i : M \rightarrow H \times E_i^\perp; \quad m \mapsto (\phi(m), p_i(m)).$$

This is a local diffeomorphism at m if and only if $D\phi(m)|_{E_i}$ is onto. Define J_i as the Radon-Nikodym derivative of Φ_i :

$$(3.2) \quad J_i(m) := \frac{\Phi_{i*}(dm)}{d_H x de'},$$

where de' is Lebesgue measure on E_i^\perp .

Let

$$(3.3) \quad \mathbf{J} = (J_1, \dots, J_N) : M \rightarrow \mathbb{R}^N.$$

By the hypothesis of the Lemma, there is some $\delta > 0$ such that the open sets $U_i := J_i^{-1}(\delta, \infty)$ cover $M\text{-supp}(a)$. Choose $\psi_i \in C_c^\infty(U_i)$ such that $\sum_i \psi_i \equiv 1$ on $M\text{-supp}(a)$. Put $a_i(x, m) := \psi_i(m)a(x, m)$, so that $A = \sum_i A_i$ where

$$(3.4) \quad A_i u(x) := \int_{U_i} a_i(x, m) u(\phi(m)^{-1}x) dm.$$

Fix $i \in \{1, \dots, N\}$. By design, Φ_i is a local diffeomorphism on $M\text{-supp}(a_i)$, so we can find a finite cover $\{V_j\}$ of $M\text{-supp}(a_i)$ by relatively compact open sets on which Φ_i is a diffeomorphism to its range. We write $\Phi_{ij} := \Phi_i|_{V_j}$ for these diffeomorphisms. Now let $\chi_j \in C^\infty(V_j)$ be a partition of unity subordinate to $\{V_j\}$ and put $a_{ij}(x, m) := \chi_j(m)a_i(x, m)$. Then

$$\begin{aligned} A_i u(x) &= \sum_j \int_{V_j} a_{ij}(x, m) u(\phi(m)^{-1}x) dm. \\ &= \sum_j \int_{(h, e') \in \Phi_i(V_j)} a_{ij}(x, \Phi_{ij}^{-1}(h, e')) u(h^{-1}x) J_i(\Phi_{ij}^{-1}(h, e'))^{-1} dh de' \\ (3.5) \quad &= \sum_j \int_{(h, e') \in \Phi_i(V_j)} k_{ij}(x, h, e') u(h^{-1}x) dh de', \end{aligned}$$

where

$$k_{ij}(x, h, e') := a_{ij}(x, \Phi_{ij}^{-1}(h, e')) J_i(\Phi_{ij}^{-1}(h, e'))^{-1}.$$

Since $J_i(m) \geq \delta$ on $M\text{-supp}(a_{ij})$, $k_{ij}(x, h, e')$ extends to a smooth compactly supported function on $\mathbf{G} \times \mathbf{H} \times E_i^\perp$. We obtain

$$A_i u(x) = \int_{\mathbf{G}} \left(\sum_j \int_{E_i^\perp} k_{ij}(x, h, e') de' \right) u(h^{-1}x) dh.$$

The quantity in parentheses is a smooth compactly supported function of $(x, h) \in \mathbf{G} \times \mathbf{H}$, so $A_i \in \Psi_c^{-\infty}(\mathcal{H})$. This completes the proof. \square

Corollary 3.3. *Any A of the form (3.1) is a multiplier of the C^* -algebra $\overline{\Psi_c^{-\infty}(\mathcal{H})}$.*

Proof. Let $K \in \Psi_c^{-\infty}(\mathcal{H})$ be given in the form (2.1), for some $k \in C_c^\infty(\mathbf{G} \times \mathbf{H})$. Then

$$AKu(x) = \int_{M \times \mathbf{H}} a(x, m) k(\phi(m)^{-1}x, h) u((\phi(m)h)^{-1}x) dm dh.$$

The map

$$M \times \mathbf{H} \rightarrow \mathbf{H}; \quad (m, h) \mapsto \phi(m)h$$

is a submersion, so by the above lemma $AK \in \Psi^{-\infty}(\mathcal{H})$. Similarly, $KA \in \Psi^{-\infty}(\mathcal{H})$. A density argument completes the proof. \square

Theorem 3.4. *Let $a \in C_c^\infty(\mathbf{G} \times M)$ and $\phi : M \rightarrow \mathbf{G}$ be smooth. If $C(\phi)$ has measure zero, then the operator A of Equation (3.1) is in $\overline{\Psi_c^{-\infty}(\mathcal{H})}$.*

Proof. Fix $\epsilon > 0$. Choose an open neighbourhood U of $C(\phi)$ with measure less than ϵ . Let χ_1, χ_2 be a smooth partition of unity on M with $\text{supp}(\chi_1) \subset U$ and $\text{supp}(\chi_2) \subset M \setminus C(\phi)$. Then $A = A_1 + A_2$ with

$$A_i u(x) := \int_M \chi_i(m) a(x, m) u(\phi(m)^{-1}x) dm.$$

Now, $A_2 \in \Psi_c^{-\infty}(\mathcal{H})$ by Lemma 3.2, and $\|A_1\| < \epsilon$ by Lemma 3.1. \square

Recall that every Lie group admits a canonical real-analytic structure. In practice, it will be real-analyticity that ensures the measure-zero critical set required for Theorem 3.4.

Corollary 3.5. *Let M be a connected real-analytic manifold, and $\phi : M \rightarrow \mathbf{H}$ be a real-analytic map with image of nonzero measure, then the operator A of (3.1) is in $\overline{\Psi_c^{-\infty}}(\mathcal{H})$.*

Proof. As above, let $d := \dim(\mathbf{H})$, $n := \dim(M)$ and $N := \binom{n}{d}$. In any analytic chart U of M , the critical set of ϕ is the zero set of the real-analytic function $\mathbf{J} : U \rightarrow \mathbb{R}^N$ of Equation (3.3). This function is not everywhere zero, by Sard's Theorem. Real-analyticity implies $\mathbf{J}^{-1}(\mathbf{0})$ has measure zero. \square

4. PRODUCTS OF LONGITUDINAL PSEUDODIFFERENTIAL OPERATORS ON LIE GROUPS

Let $\mathbf{H}_1, \dots, \mathbf{H}_r$ be connected Lie subgroups (not necessarily closed) of the connected Lie group \mathbf{G} . Let \mathbf{H} denote the subgroup they generate:

$$\mathbf{H} := \{x_1 x_2 \cdots x_k \mid \text{Each } x_i \text{ is in some } \mathbf{H}_j\}.$$

This is a connected Lie subgroup whose Lie algebra \mathfrak{h} is the Lie algebra generated by $\mathfrak{h}_1, \dots, \mathfrak{h}_r$. We use \mathcal{H} (resp. \mathcal{H}_j) to denote the foliation of \mathbf{G} by left-cosets of \mathbf{H} (resp. \mathbf{H}_j).

Theorem 4.1. *With the above notation,*

$$\overline{\Psi_c^{-\infty}}(\mathcal{H}_1) \cdots \overline{\Psi_c^{-\infty}}(\mathcal{H}_r) \subseteq \overline{\Psi_c^{-\infty}}(\mathcal{H})$$

Proof. Let $\mathbf{H}_1 \mathbf{H}_2 \cdots \mathbf{H}_r$ denote the set of products $\{h_1 h_2 \cdots h_r \mid h_j \in \mathbf{H}_j \text{ for all } j\}$. We first prove the theorem under the assumption that $\mathbf{H}_1 \mathbf{H}_2 \cdots \mathbf{H}_r$ has nonzero measure in \mathbf{H} .

Let $K_j = \text{Op}_{\mathbf{H}}(k_j) \in \Psi_c^{-\infty}(\mathcal{H}_j)$, with $k_j \in C_c^\infty(\mathbf{G} \times \mathbf{H})$. By iterating equation (2.1), we see that

$$K_1 K_2 \cdots K_r u(x) = \int_{\mathbf{H}_1 \times \cdots \times \mathbf{H}_r} a(x, h_1, \dots, h_r) u((h_1 \cdots h_r)^{-1}x) dh_1 \cdots dh_r,$$

where

$$a(x, h_1, \dots, h_r) := \prod_{j=1}^r k_j((h_1 \cdots h_{j-1})^{-1}x, h_j).$$

Here, $a \in C_c^\infty(\mathbf{G} \times (\mathbf{H}_1 \times \cdots \times \mathbf{H}_r))$ and the map $(h_1, \dots, h_r) \mapsto h_1 \cdots h_r$ is real-analytic, so Corollary 3.5 implies that $K_1 \cdots K_r \in \overline{\Psi_c^{-\infty}}(\mathcal{H})$.

Now we drop the assumption on the product $\mathbf{H}_1 \cdots \mathbf{H}_r$. Note that $K_1 \cdots K_r$ is in the multiplier algebra of $\overline{\Psi_c^{-\infty}}(\mathcal{H})$, by Corollary 3.3. Since $\mathbf{H}_1, \dots, \mathbf{H}_r$ generate \mathbf{G} , there is some $n \in \mathbb{N}$ for which the set

$$\underbrace{(\mathbf{H}_1 \mathbf{H}_2 \cdots \mathbf{H}_r \mathbf{H}_r \cdots \mathbf{H}_2 \mathbf{H}_1)(\mathbf{H}_1 \mathbf{H}_2 \cdots \mathbf{H}_r \mathbf{H}_r \cdots \mathbf{H}_2 \mathbf{H}_1) \cdots (\mathbf{H}_1 \mathbf{H}_2 \cdots \mathbf{H}_r \mathbf{H}_r \cdots \mathbf{H}_2 \mathbf{H}_1)}_{n \text{ copies}}$$

has positive measure in G . By employing the formula (2.2) for the adjoint of K_j , the previous argument shows that $(K_1 \cdots K_r K_r^* \cdots K_1^*)^n \in \overline{\Psi_c^{-\infty}(\mathcal{H})}$. But now standard C^* -algebra theory implies that $K_1 \cdots K_r$ is in $\overline{\Psi_c^{-\infty}(\mathcal{H})}$ (see [Dav96, I.5.3]).

□

5. LOCALLY HOMOGENEOUS STRUCTURES

We now pass to manifolds which are locally modelled on Lie groups. We continue with the above notation: \mathfrak{g} is the Lie algebra of a connected Lie group G ; $\mathfrak{h}_1, \dots, \mathfrak{h}_r$ is a family of Lie subalgebras and \mathfrak{h} is the Lie algebra they generate; $\mathcal{H}_1, \dots, \mathcal{H}_r$ and \mathcal{H} are the corresponding left-coset foliations.

Recall (Definition 1.1) that a family of foliations $\mathcal{F}_1, \dots, \mathcal{F}_r$ of a manifold \mathcal{X} is called *locally homogeneous* if \mathcal{X} admits an atlas of local diffeomorphisms from G , under which the fibrations map to $\mathcal{H}_1, \dots, \mathcal{H}_r$. As described in the introduction, such a family $\mathcal{F}_1, \dots, \mathcal{F}_r$ generates a foliation $\mathcal{F} \subseteq T\mathcal{X}$, which in each chart is \mathcal{H} .

Theorem 1.2 is then an immediate consequence of Theorem 4.1 via a partition of unity argument.

We now apply this to the key example of generalized flag varieties.

Example 5.1. Let G be a complex semisimple Lie group of rank r . Fix a Cartan subalgebra \mathfrak{h} . Fix a system of positive roots Σ^+ , with simple roots Π . Let $\mathfrak{n} := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$, $\bar{\mathfrak{n}} := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha}$, and let $\bar{\mathfrak{b}} := \mathfrak{h} \oplus \bar{\mathfrak{n}}$ be the ‘lower’ Borel subalgebra. The corresponding Lie groups will be given by upper case letters.

Let $\mathcal{X} := G/\bar{B}$ be the flag variety of G . For any $S \subseteq \Pi$, let $\langle S \rangle$ denote the set of positive roots spanned by S , ie,

$$\langle S \rangle := \{ \alpha \in \Sigma^+ \mid \alpha = \sum_{\beta \in S} n_\beta \beta \text{ for some } n_\beta \in \mathbb{N} \}.$$

Let $\mathfrak{n}_S := \bigoplus_{\alpha \in \langle S \rangle} \mathfrak{g}_\alpha$. Let $\bar{\mathfrak{p}}_S$ denote the parabolic subalgebra $\bar{\mathfrak{b}} \oplus \mathfrak{n}_S$. Let $\mathcal{Y}_S := G/\bar{P}_S$ be the corresponding partial flag variety. Let \mathcal{F}_S denote the fibration of \mathcal{X} by fibres of the quotient map $\tau_S : \mathcal{X} \rightarrow \mathcal{Y}_S$.

The map $\varphi : N \hookrightarrow G \twoheadrightarrow G/\bar{B} = \mathcal{X}$ is a diffeomorphism onto its range. It is clearly N -equivariant. With this as a chart, the fibres of τ_S pull back to the left cosets of $N \cap \bar{P}_S = N_S$. By taking G -translates, we can cover \mathcal{X} with such charts. Thus we see that the family of fibrations $(\mathcal{F}_S)_{S \subseteq \Pi}$ is locally homogeneous with structural data $(\mathfrak{n}_S)_{S \subseteq \Pi} \leq \mathfrak{n}$.

Theorem 1.2 yields the following statement. Let $S_1, \dots, S_r \subseteq \Pi$ and put $S = \bigcup_{i=1}^r S_i$. Then

$$\overline{\Psi_c^{-\infty}(\mathcal{F}_{S_1})} \cdots \overline{\Psi_c^{-\infty}(\mathcal{F}_{S_r})} \subseteq \overline{\Psi_c^{-\infty}(\mathcal{F}_S)}.$$

In particular, if A_α is a longitudinal pseudodifferential operator along $\mathcal{F}_{\{\alpha\}}$, for each $\alpha \in \Pi$, then their product is a compact operator. This is the generalization of [Yun10, Theorem 1.3] for arbitrary flag manifolds.

6. ESSENTIAL ORTHOGONALITY OF SUB-TYPES

We conclude with a brief application of the above results in noncommutative harmonic analysis for compact groups.

Let K be a compact Lie group, and let K_1 and K_2 be closed subgroups. Let $U : K \rightarrow \mathcal{B}(H)$ be a unitary representation of K with finite multiplicities. What can be said of the relative position of the K_1 - and K_2 -invariant subspaces of H , or more generally of the isotypical subspaces? This question arises naturally in harmonic analysis on flag varieties (see [Yun10]).

Let us introduce some notation. If π is an irreducible representation of a subgroup K' of K , p_π will denote the orthogonal projection onto H_π , the π -isotypical subspace of $U|_{K'}$. If $S \subseteq \hat{K}'$ is a set of K' -types, we put $p_S := \sum_{\pi \in S} p_\pi$. Note that if $\sigma \in \hat{K}$ and $\pi \in \hat{K}'$, then p_σ and p_π commute.

We define the *inner product* of subspaces $H_1, H_2 \leq H$ by

$$\langle H_1, H_2 \rangle := \sup\{\langle \xi_1, \xi_2 \rangle \mid \xi_j \in H_j, \|\xi_j\| \leq 1\}.$$

Suppose to begin with that $K = K_1 \times K_2$. If the representation H has finite K -multiplicities, the isotypical subspaces H_{π_1} and H_{π_2} will have finite dimensional intersection—namely, $H_{\pi_1 \otimes \pi_2}$ —and moreover will be ‘perpendicular’ in the sense that their respective orthocomplements $H_{\pi_1} \cap (H_{\pi_2})^\perp$ and $H_{\pi_2} \cap (H_{\pi_1})^\perp$ are orthogonal.

Clearly this is not true in generality. For instance, let $K = \mathrm{SU}(3)$ and K_1 and K_2 be the subgroups obtained by embedding $\mathrm{SU}(2)$ in the upper-left and lower-right corners of K , respectively. There are infinitely many irreducible $\mathrm{SU}(3)$ -representations which contain both a nonzero K_1 -fixed vector and a nonzero K_2 -fixed vector, and these fixed vectors are not in general orthogonal. However, they are asymptotically orthogonal, in the sense that for any $\epsilon > 0$, there are only finitely many K -types in which the K_1 - and K_2 -fixed subspaces have inner product greater than ϵ .

With this example in mind we make the following definition. We use V^σ to denote the vector space underlying an irreducible representation $\sigma \in \hat{K}$.

Definition 6.1. We say the subgroups K_1 and K_2 of K are *essentially orthotypical* if, for any $\pi_1 \in \hat{K}_1$, $\pi_2 \in \hat{K}_2$ and $\epsilon > 0$, there are only finitely many K -types $\sigma \in \hat{K}$ for which $\langle (V^\sigma)_{\pi_1}, (V^\sigma)_{\pi_2} \rangle \geq \epsilon$.

An equivalent definition is given by the following lemma.

Lemma 6.2. *Let K_1 and K_2 be closed subgroups of a compact Lie group K . The following are equivalent:*

- (i) K_1 and K_2 are essentially orthotypical.
- (ii) For any $\pi_1 \in \hat{K}_1$ and $\pi_2 \in \hat{K}_2$, $p_{\pi_1} p_{\pi_2}$ is a compact operator on every unitary K -representation with finite multiplicities.

Proof. (i) \Rightarrow (ii): Let S be the set of $\sigma \in \hat{K}$ for which $\langle (V^\sigma)_{\pi_1}, (V^\sigma)_{\pi_2} \rangle \geq \epsilon$. Then on V^σ for any $\sigma \notin S$, $\|p_{\pi_1} p_{\pi_2}\| < \epsilon$. Therefore

$$p_{\pi_1} p_{\pi_2} = p_S p_{\pi_1} p_{\pi_2} + p_S^\perp p_{\pi_1} p_{\pi_2}$$

with the right-hand side being the sum of a finite rank operator and an operator of norm at most ϵ .

(ii) \Rightarrow (i): Let $\epsilon > 0$. Fix any enumeration $\{\sigma_1, \sigma_2, \dots\}$ of \hat{K} , and let $S_n := \{\sigma_1, \dots, \sigma_n\}$. Put $H := \bigoplus_{n=1}^\infty V^{\sigma_n}$. The projections p_{S_n} on H converge strongly to 1 as $n \rightarrow \infty$, so by the compactness of $p_{\pi_1} p_{\pi_2}$ we have $\|(p_{\pi_1} p_{\pi_2}) p_{S_n}^\perp\| < \epsilon$ for sufficiently large n .

Now let σ be any K -type not in S_n and suppose $\xi_j \in (V^\sigma)_{\pi_j}$ for $j = 1, 2$, with $\|\xi_j\| \leq 1$. After including V^σ into H , we have

$$|\langle \xi_1, \xi_2 \rangle| = |\langle p_{S_n}^\perp p_{\pi_1} \xi_1, p_{S_n}^\perp p_{\pi_2} \xi_2 \rangle| = |\langle \xi_1, p_{S_n}^\perp p_{\pi_1} p_{\pi_2} \xi_2 \rangle| < \epsilon.$$

Thus $\langle (V^\sigma)_{\pi_1}, (V^\sigma)_{\pi_2} \rangle < \epsilon$ for all $\sigma \notin S_n$. □

Remark 6.3. The representation $H = \bigoplus_{n=1}^\infty V^{\sigma_n}$ used in the proof of (ii) \Rightarrow (i) could be replaced by any representation which contains every K -type—for instance, the regular representation.

Theorem 6.4. *Let K_1, K_2 be closed subgroups of a compact Lie group K . If K_1, K_2 generate K then they are essentially orthotypical.*

Proof. Fix $\pi_1 \in \hat{K}_1, \pi_2 \in \hat{K}_2$ and $\epsilon > 0$. Let $\chi_{\pi_j} \in C^\infty(K_j)$ denote the character of π_j . Let U be the left-regular representation of K on $L^2(K)$. By the orthogonality of characters,

$$p_{\pi_j} f(x) = \int_{K_j} \overline{\chi_{\pi_j}(x)} f(k^{-1}x) dx$$

for any $f \in L^2(K)$. That is, p_{π_j} is a longitudinally smoothing operator for the coset fibration of $K_j \leq K$. Thus, using Remark ??, Lemma 6.2 gives the result. \square

It is natural to ask about the converse of Theorem 6.4. We will see that the converse holds at least if K is compact semisimple.

In fact, if the subgroup generated by K_1 and K_2 is not dense in K then K_1 and K_2 are not essentially orthotypical. For consider the representation of K on $L^2(K/K')$, where K' is the closed subgroup generated by K_1 and K_2 . The Peter-Weyl Theorem gives the decomposition

$$L^2(K/K') \cong \bigoplus_{\sigma \in \hat{K}} V^\sigma \otimes (V^{\sigma*})_{\pi_0},$$

where π_0 is the trivial representation of K' . (On the summands of the right hand side, the K -representation is by $\sigma \otimes I$.) Thus $L^2(K/K')$ has finite K -multiplicities. It also has an infinite dimensional K' -fixed subspace, since V^σ contains a K' -fixed vector if and only if $V^{\sigma*}$ does. The projections onto the trivial K_j -types (for $j = 1, 2$) both act as the identity on this subspace, so their product cannot be compact.

Compact semisimple groups have no connected dense subgroups ([MV93]), which gives the converse to Theorem 6.4 for K semisimple.

REFERENCES

- [AS71] M. F. Atiyah and I. M. Singer. The index of elliptic operators. IV. *Ann. of Math. (2)*, 93:119–138, 1971.
- [Con82] A. Connes. A survey of foliations and operator algebras. In *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, volume 38 of *Proc. Sympos. Pure Math.*, pages 521–628. Amer. Math. Soc., Providence, R.I., 1982.
- [Con94] Alain Connes. *Noncommutative geometry*. Academic Press Inc., San Diego, CA, 1994.
- [ČSS01] Andreas Čap, Jan Slovák, and Vladimír Souček. Bernstein-Gelfand-Gelfand sequences. *Ann. of Math. (2)*, 154(1):97–113, 2001.
- [Dav96] Kenneth R. Davidson. *C^* -algebras by example*, volume 6 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1996.
- [JK95] P. Julg and G. Kasparov. Operator K -theory for the group $SU(n, 1)$. *J. Reine Angew. Math.*, 463:99–152, 1995.
- [Jul02] Pierre Julg. La conjecture de Baum-Connes à coefficients pour le groupe $Sp(n, 1)$. *C. R. Math. Acad. Sci. Paris*, 334(7):533–538, 2002.
- [Kas84] G. G. Kasparov. Lorentz groups: K -theory of unitary representations and crossed products. *Dokl. Akad. Nauk SSSR*, 275(3):541–545, 1984.
- [Laf02] V. Lafforgue. K -théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes. *Invent. Math.*, 149:1–95, 2002.
- [MV93] Enrique Macías Virgós. Nonclosed Lie subgroups of Lie groups. *Ann. Global Anal. Geom.*, 11(1):35–40, 1993.
- [Roe87] John Roe. Finite propagation speed and Connes’ foliation algebra. *Math. Proc. Cambridge Philos. Soc.*, 102(3):459–466, 1987.
- [Tay84] Michael E. Taylor. Noncommutative microlocal analysis. I. *Mem. Amer. Math. Soc.*, 52(313):iv+182, 1984. Preprint of 2ed online at <http://www.math.unc.edu/Faculty/met/ncmlms.pdf>.
- [Yun10] Robert Yuncken. Products of longitudinal pseudodifferential operators on flag varieties. *J. Funct. Anal.*, 258(4):1140–1166, 2010.

- [Yun11] Robert Yuncken. The Bernstein-Gelfand-Gelfand complex and Kasparov Theory: the case of $\mathrm{SL}(3, \mathbb{C})$. *Adv. Math.*, 226:1474–1512, 2011.
E-mail address: `yuncken@math.univ-bpclermont.fr`